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The rational homotopy Lie algebra of classifying spaces for formal two-stage spaces

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Abstract

We compute the center and nilpotency of the graded Lie algebra $\pi_*(\Omega \mathrm{Baut}_1(X)) \otimes \mathbb{Q}$ for a large class of formal spaces X. The latter calculation determines the rational homotopical nilpotency of the space of self-equivalences $\mathrm{aut}_1(X)$ for these X. Our results apply, in particular, when X is a complex or symplectic flag manifold.

MSC: Primary 55P62; 55R35

1. Introduction

Given a CW complex X, let $\mathrm{aut}_1(X)$ denote the identity component of the space of self-equivalences of X and $\mathrm{Baut}_1(X)$ the classifying space for this topological monoid [3]. Recall that $\mathrm{Baut}_1(X)$ classifies orientable fibrations with fibre X [1,16].

In this paper, we describe the structure of the rational homotopy Lie algebra of the classifying space $\mathrm{Baut}_1(X)$ when X is a formal space with a two-stage Sullivan minimal model. We compare the calculation of $\pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q}$ for two-stage X in [13] – which gives the underlying vector space – with Sullivan's differential graded Lie algebra model for $\mathrm{Baut}_1(X)$ [17], which gives the Lie structure. Our most general result is the identification of cycle representatives in Sullivan's model for homotopy elements of $\pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q}$. Using this result and confirmed cases of a famous conjecture of Halperin, we compute the center and nilpotency of $\pi_*(\Omega \mathrm{Baut}_1(X)) \otimes \mathbb{Q}$ for a large class of pure, formal spaces X.

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Throughout this paper, we assume all spaces X are simply connected complexes of finite type with $\dim(\pi_*(X)\otimes\mathbb{Q}) < +\infty$. This ensures that the rationalization of $\mathrm{aut}_1(X)$ is a nilpotent H-group. A basic problem then is to compute the *rational homotopical nilpotency*, $\mathrm{Hnil}_0(\mathrm{aut}_1(X))$, which is defined to be the length of the longest rationally essential commutator in $\mathrm{aut}_1(X)$. Our results include the following new calculations of this invariant.

1.1. Examples

Given any finite-dimensional graded vector space V, let $\min(V) = \min\{n \mid V^n \neq 0\}$ and $\max(V) = \max\{n \mid V^n \neq 0\}$. Given any simply connected graded algebra A^*, let $Q^*(A^*) = A^*/A^+ \cdot A^+$ denote the graded vector space of indecomposables of A^*.

Let $S = (n_1, \dots, n_k)$ be a nondecreasing sequence of positive integers. A finite (possibly empty) sequence T of the letters n_i is *admissable* if repetitions of a letter n_i occur only when n_i is even. Put $\sum(T) = \sum_{n_i \in T} n_i$; the empty sum is defined to be zero. Define $S_1, \dots, S_m$ to be an *elementary sequence* if (i) each S_j is nonempty and admissable, (ii) S_{j+1} is obtained from S_j by replacing one occurance of a letter $n_i \in S_j$ by an admissable sequence T with $\sum(T) < n_i$ and (iii) $S_1 = (n_k)$. Define $\mathrm{maxlen}\,(S)$ to be the length of the longest elementary sequence $S_1, \dots, S_m$ obtainable from S. Finally, given a finite-dimensional graded vector space V, write $V = \mathbb{Q}(x_1, \dots x_k)$, where the x_i are homogeneous of nondecreasing degree, and let $\mathrm{maxlen}\,(V) = \mathrm{maxlen}\,(S)$, where $S = (|x_1|, \dots, |x_k|)$.

(1) Let $S = S^{2m_1} \times \cdots \times S^{2m_k} \times S^{2n_1+1} \times \cdots \times S^{2n_l+1}$, be a product of spheres, with $1 \leq m_1 \leq \cdots \leq m_k$ and $1 \leq n_1 \leq \cdots \leq n_l$. Let $c = \mathrm{maxlen}\,(S)$, where $S = (2n_1+1, \dots, 2n_l+1)$. Then

$$\mathrm{Hnil}_0(\mathrm{aut}_1(S)) = \begin{cases} 1 & \text{if } c = 0, \\ c & \text{if } 2m_k \leq n_l + 1, \\ c+1 & \text{if } 2m_k > n_l + 1. \end{cases}$$

(2) More generally, suppose F is any space with $H^*(F,\mathbb{Q})$ a finite tensor product of polynomial, truncated polynomial and exterior algebras. Let $K \subseteq H^*(F,\mathbb{Q})$ be a maximal free graded subalgebra and let $c = \mathrm{maxlen}\,(Q^*(K))$. Then

$$\mathrm{Hnil}_0(\mathrm{aut}_1(F)) = \begin{cases} 1 & c = 0, \\ c & \text{if } \max(\pi_*(F) \otimes \mathbb{Q}) \leq \max(Q^*(K)), \\ c+1 & \text{if } \max(\pi_*(F) \otimes \mathbb{Q}) > \max(Q^*(K)). \end{cases}$$

(3) Let $F = G/H$, where $G = U(n)$ (respectively, $G = Sp(n)$) and $H = U(n_1) \times \cdots \times U(n_k)$ (respectively, $H = Sp(n_1) \times \cdots \times Sp(n_k)$). Let $m = n_1 + \cdots + n_k$. Then $\mathrm{Hnil}_0(\mathrm{aut}_1(F)) = \max\{n - m, 1\}$.

(4) Let F be any pure, elliptic, formal space (see Section 2, for definitions) with $\dim(Q^{\mathrm{even}}(H^*(F,\mathbb{Q}))) \le 3$. Let $c = \operatorname{maxlen}(Q^{\mathrm{odd}}(H^*(F,\mathbb{Q})))$. Then

$$\mathrm{Hnil}_0(\mathrm{aut}_1(F)) = \begin{cases} 1 & \text{if } c = 0, \\ c & \text{if } \max(\pi_*(F)\otimes Q) \le \max\,(Q^{\mathrm{odd}}(H^*(F,\mathbb{Q}))), \\ c+1 & \text{if } \max\,(\pi_*(F)\otimes Q) > \max(Q^{\mathrm{odd}}(H^*(F,\mathbb{Q}))). \end{cases}$$

2. Two-stage spaces and Halperin's conjecture

For the remainder of the paper, we assume all groups are rational.

By a *two-stage* space X, we mean one whose rationalization $X_{\mathbb{Q}}$ appears as the total space in a principal fibration of the form $K_1 \hookrightarrow X_{\mathbb{Q}} \xrightarrow{q} K_0$, where $K_i = \prod_n K(V_i^n, n)$ for some finite-dimensional graded rational vector spaces V_i, $i=0,1$. The Sullivan minimal model $(\mathscr{M}_X, d_X)$ for a two-stage space X is a two-stage differential graded algebra (dga). That is, $\mathscr{M}_X = \Lambda(V_0)\otimes_{d_X}\Lambda(V_1)$, with $d_X(V_0)=0$ and $d_X(V_1)\subseteq\Lambda(V_0)$. We may assume $d_X : V_1 \to \Lambda(V_0)$ is an injection. Fixing bases, write $V_0 = \mathbb{Q}(x_1,\ldots,x_m)$ and $V_1 = \mathbb{Q}(y_1,\ldots,y_n)$ so that $d_X(x_i)=0$ and $d_X(y_j) = R_j(x_1,\ldots,x_m)$, a polynomial without linear term in the x_i. A two-stage space is *pure* if V_1 is oddly graded and $d_X(V_1)\subseteq\Lambda(V_0^{\mathrm{even}})$. Homogeneous spaces of a compact Lie group by a closed subgroup are pure by Borel [2].

If X is formal with finite-dimensional rational homotopy then X is a two-stage space [4]. In fact, a two-stage space X is formal if and only if the sequence $R_1, R_2, \ldots, R_n$ forms a regular sequence in the free algebra $\Lambda(V_0)$. An important class of examples are the F_0-*spaces*, by which we mean spaces X with finite-dimensional cohomology and homotopy (*elliptic spaces*) such that $H^{\mathrm{odd}}(X)=0$. The class includes quotients G/H of a compact, connected Lie group by a closed subgroup of maximal rank [2]. Halperin proved F_0-spaces are formal in [6], where he also made the following conjecture:

Halperin's Conjecture. *The rational Serre spectral sequence collapses at the E_2-term for every $\mathbb{Q}$-orientable fibration of the form $X \hookrightarrow E \to B$ with X an F_0-space.*

Halperin's conjecture has been confirmed for the homogeneous spaces mentioned above [12] and in several other special cases [8,7].

In [9, Theorem A], Meier showed that Halperin's conjecture is equivalent to the assertion that $\pi_{\mathrm{even}}(\mathrm{aut}_1(X)) = 0$ for all F_0-spaces. His result implies

Theorem 2.1 (Meier). *Let X be an F_0-space satisfying Halperin's conjecture. Then* $\mathrm{Baut}_1(X)$ *is a rational H-space and* $\mathrm{Hnil}_0(\mathrm{aut}_1(X)) = 1$.

Proof. Of course, the second assertion is a consequence of the first. For the first, note that by Meier's result, $\pi_{\mathrm{odd}}(\mathrm{Baut}_1(X))=0$. Now observe that any space with only even rational homotopy is rationally an H-space. □

3. Rational homotopy of the space of self-equivalences

We describe the graded vector space $\pi_*(\Omega\mathrm{Baut}_1(X)) = \pi_*(\mathrm{aut}_1(X))$ when X is a two-stage space. With notation as in Section 2, define graded spaces $L_0(X)$ and $L_1(X)$ by setting

$$L_i^n(X) = \bigoplus_{k\geq 0} H^k(X)\otimes V_i^{n+k},$$

where $n > 0$ when $i = 0$ and $n \geq 0$ when $i = 1$. Thus, for example, $L_0(X)$ is spanned by elements of the form $\alpha\otimes x_i$ where $\alpha\in H^*(X)$ is homogeneous of degree strictly less than $|x_i|$. The degree of $\alpha\otimes x_i$ in $L_0(X)$ is then the difference $|x_i| - |\alpha|$. Let $L(X) = L_0(X)\oplus L_1(X)$.

Define a linear map $D : L_0(X)\to L_1(X)$ of degree -1 by

$$D(\alpha\otimes x_i) = \sum_{j=1}^{n} \alpha\left\{\frac{\partial R_j}{\partial x_i}\right\}\otimes y_j, \tag{1}$$

where we write $\{P\}$ to denote the cohomology class in $H^*(X)$ represented by an element $P\in\mathscr{M}_X$. Set

$$\mathscr{L}_0(X) = \ker\{D : L_0(X)\to L_1(X)\} \quad\text{and}\quad \mathscr{L}_1(X) = \mathrm{cok}\{D : L_0(X)\to L_1(X)\},$$

where we force $\mathscr{L}_1(X)$ to be connected by eliminating the elements of degree zero. We then have

Theorem 3.1. *Let X be any two-stage space. As graded vector space*

$$\pi_*(\Omega\mathrm{Baut}_1(X)) = \mathscr{L}_0(X)\oplus\mathscr{L}_1(X).$$

Proof. The principal fibration for $X_{\mathbb{Q}}$ determines a fibration on function spaces

$$\mathrm{map}(X_{\mathbb{Q}}, K_1; 0)\xrightarrow{\Phi}\mathrm{aut}_1(X_{\mathbb{Q}})\xrightarrow{q}\mathrm{map}(X_{\mathbb{Q}}, K_0; q).$$

By Thom's classical result [18], $\pi_*(\mathrm{map}(X_{\mathbb{Q}}, K_i; f)) \cong L_i(X)$, $i = 0, 1$, $f = q, 0$. We thus have a long exact sequence

$$\cdots\to L_0^{n+1}(X)\xrightarrow{\partial_0} L_1^n(X)\xrightarrow{\Phi_*}\pi_n(\mathrm{aut}_1(X))\to L_0^n(X)\xrightarrow{\partial_0} L_1^{n-1}(X)\to\cdots .$$

In [13, Chapter 6], we describe ∂_0 for general maps $f : X\to X$. When f is the identity our result shows ∂_0 coincides with D (cf. [14, Lemma 4.4]). $\square$

In Section 4, we determine the Lie structure on the spaces $\mathscr{L}_i(X)$ $i = 0, 1$. We conclude this section with two simple consequences of Meier's result.

Theorem 3.2. *Let X be an F_0-space. Then X satisfies Halperin's conjecture if and only if $\pi_*(\Omega\mathrm{Baut}_1(X)) = \mathscr{L}_1(X)$.*

Proof. Note that $\mathscr{L}_0(X)$ is evenly graded. Thus [9, Theorem A] implies X satisfies Halperin's conjecture if and only if $\mathscr{L}_0(X) = 0$. $\square$

Let Y be another two-stage space with Sullivan minimal model $\mathscr{M}_Y = \Lambda(W_0) \otimes_{d_Y} \Lambda(W_1)$. For $i = 0, 1$, let $L_i(X \times Y, Y) \subseteq L_i(X \times Y)$ denote the subspace given in degree n by

$$L_i^n(X \times Y, Y) = \bigoplus_{k \geq 0} H^k(X \times Y) \otimes W_i^{k+n}.$$

We prove

Theorem 3.3. *Let X be an F_0-space satisfying Halperin's conjecture and Y any two-stage space. Then*

$$\mathscr{L}_0(X \times Y) = \ker\{D : L_0(X \times Y, Y) \to L_1(X \times Y, Y)\}.$$

Proof. Since X satisfies Halperin's conjecture, $D(\alpha \otimes x_i) \neq 0$ for all $\alpha \in H^{<|x_i|}(X)$ and all x_i in our basis for V_0. By the rational Künneth Theorem and the definition of D it follows that $D(\alpha \otimes x_i) \neq 0$ when $\alpha \in H^{<|x_i|}(X \times Y)$, as well. $\square$

4. Cycle representatives in Sullivan's model for $\mathrm{Baut}_1(X)$

We next consider Sullivan's differential graded Lie algebra (dgla) model for $\mathrm{Baut}_1(X)$ as described in [17, Section 11]. For $n > 1$, let $\mathrm{Der}_n(\mathscr{M}_X)$ denote the space of derivations of degree $-n$ of the graded algebra $\mathscr{M}_X$. That is, $\mathrm{Der}_n(\mathscr{M}_X)$ consists of maps $\theta : \mathscr{M}_X \to \mathscr{M}_X$ that lower degrees by n and satisfy $\theta(xy) = \theta(x)y + (-1)^{n|x|}x\theta(y)$. For $n = 1$, we require, additionally, that θ commute with the differential d_X. The Lie bracket of two derivations is the graded commutator: $[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - (-1)^{|\theta_1||\theta_2|}\theta_2 \circ \theta_1$. Define a differential ∂_X by $\partial_X(\theta) = [d_X, \theta]$. The pair $(\mathrm{Der}_+(\mathscr{M}_X), \partial_X)$ is then a dgla model for the rational homotopy of $\mathrm{Baut}_1(X)$. (For a proof, use [5, Theorem 2] together with [11].) In particular, $H(\mathrm{Der}_+(\mathscr{M}_X), \partial_X) \cong \pi_*(\Omega \mathrm{Baut}_1(X))$, as graded Lie algebras.

A basis for $\mathrm{Der}_+(\mathscr{M}_X)$ as graded space can be obtained using *elementary derivations*. Suppose X is two-stage with notation as in Section 2. Then, given $z_k \in \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ and $P \in \mathscr{M}_X$ homogeneous with degree $|P| < |z_k|$, let $P\partial z_k$ denote the derivation carrying z_k to P and vanishing on the other basis elements of $V_0 \oplus V_1$. Observe that ∂_X is given by

$$\partial_X(P\partial z_k) = d_X(P)\partial z_k - (-1)^{|z_k|-|P|} \sum_{j=1}^{n} P \frac{\partial R_j}{\partial z_k} \partial y_k. \tag{2}$$

If X is formal, there is a dga map $\rho : (\mathscr{M}_X, d_X) \to (H^*(X), 0)$ inducing a homology isomorphism. The map of graded spaces $p : \mathrm{Der}_+(\mathscr{M}_X) \to L(X)$ defined by $p(P\partial z_k) = \rho(P) \otimes z_k$ is then a surjection. The map p induces an isomorphism on homology between $(\mathrm{Der}_+(\mathscr{M}_X), \partial_X)$ and $(L(X), D)$ as dg vector spaces. We look for the actual cycle representatives in $\mathrm{Der}_+(\mathscr{M}_X)$ for the subspaces $\mathscr{L}_0(X)$ and $\mathscr{L}_1(X)$.

Define subspaces $D_0(X), D_1(X)$ and $B(X)$ of $\mathrm{Der}_+(\mathscr{M}_X)$ by

$$D_0(X) = \mathrm{Span}\{P\partial x_i \mid P \in \Lambda(V_0), |P| < |x_i|,\ i \in \{1,\dots,m\}\},$$
$$D_1(X) = \mathrm{Span}\{P\partial y_j \mid P \in \Lambda(V_0), |P| < |y_j|,\ j \in \{1,\dots,n\}\}$$

and

$$B(X) = \mathrm{Span}\{P y_k \partial y_j \mid P \in \Lambda(V_0), |P| + |y_k| < |y_j|,\ j,k \in \{1,\dots,n\}\}.$$

Of course, in degree 1 we must restrict to the kernel of ∂_X.

Note $D_0(X)$ and $D_1(X)$ are sub Lie algebras and $B(X) \oplus D_0(X) \oplus D_1(X)$ a sub-dgla of $(\mathrm{Der}_+(\mathscr{M}_X), \partial_X)$. Since we have restricted to $P \in \Lambda(V_0)$, we have $\partial_X(D_1(X)) = 0$ and $\partial_X(B(X) \oplus D_0(X)) \subseteq D_1(X)$. Also $p(B(X)) = 0$. We prove

Theorem 4.1. *The diagram*

$$\begin{tikzcd} B(X) \oplus D_0(X) \arrow[r, "\partial_X"] \arrow[d, "p"] & D_1(X) \arrow[d, "p"] \\ L_0(X) \arrow[r, "D"] & L_1(X) \end{tikzcd}$$

commutes up to sign. The vertical maps are surjections.

Proof. Since X is hyperformal, $\Lambda(V_0)$ contains all cocycle representatives for $H^*(X)$. Commutativity up to sign follows from (1) and (2) above and the formality of X: the map ρ chooses cohomology classes multiplicatively. $\square$

Set

$$\mathscr{D}_0(X) = \ker\{\partial_X : B(X) \oplus D_0(X) \to D_1(X)\}$$

and

$$\mathscr{D}_1(X) = \mathrm{cok}\{\partial_X : B(X) \oplus D_0(X) \to D_1(X)\}.$$

We may naturally view $\mathscr{D}_1(X)$ as a subspace of $\mathrm{Der}_+(\mathscr{M}_X)$ by taking vector space complements in each degree. Our main result in this section is

Theorem 4.2. *If X is formal two-stage, the subspaces $\mathscr{D}_0(X)$ and $\mathscr{D}_1(X)$ contain all cycle representatives in* $\mathrm{Der}_+(\mathscr{M}_X)$ *for the subspaces $\mathscr{L}_0(X)$ and $\mathscr{L}_1(X)$ of* $\pi_*(\Omega\mathrm{Baut}_1(X))$, *respectively.*

Proof. We break the proof into four lemmas. We first prove that $p : \mathrm{Der}_+(\mathscr{M}_X) \to L(X)$ induces surjections $p : \mathscr{D}_i(X) \to \mathscr{L}_i(X),\ i = 0, 1$.

Lemma 4.3. *The map p induces a bijection $p : \mathscr{D}_1(X) \to \mathscr{L}_1(X)$.*

Proof. This is just a diagram chase. $\square$

Lemma 4.4. *The map p restricts to a surjection $p : \mathscr{D}_0(X) \to \mathscr{L}_0(X)$.*

Proof. Suppose $z = \sum_{i=1}^{m} \alpha_i \otimes x_i \in \mathscr{L}_0(X)$ so that $D(z) = 0$. Choose cocycle representatives $P_i \in \Lambda(V_0)$ for the α_i and let $\theta = \sum_{i=1}^{m} P_i \partial x_i$. By commutativity of the diagram,

$$0 = p(\partial_X(\theta)) = p\left(\sum_{i=1}^{m}\sum_{j=1}^{n} P_i \frac{\partial R_j}{\partial x_i}\partial y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} \rho\left(P_i \frac{\partial R_j}{\partial x_i}\right) \otimes y_j.$$

The regularity of $\{R_1, \ldots, R_n\}$ in $\Lambda(V_0)$ implies

$$\sum_{i=1}^{m} P_i \frac{\partial R_j}{\partial x_i} = \sum_{k=1}^{n} Q_{jk} R_k$$

for each $j = 1, \ldots, n$ and some $Q_{jk} \in \Lambda(V_0)$. Set $\phi = \sum_{j=1}^{n}\sum_{k=1}^{n} Q_{jk} y_k \partial y_j \in B(X)$. Then $\partial_X(\theta - \phi) = 0$ and $p(\theta - \phi) = z$. $\square$

We next show that the nontrivial boundaries in $D_1(X)$ are precisely $\partial_X(B(X) \oplus D_0(X))$.

Lemma 4.5. *Let $\theta \in \mathrm{Der}_+(\mathscr{M}_X)$. If $\partial_X(\theta) \in D_1(X)$, then $\theta = \phi + \xi$, where $\phi \in B(X) \oplus D_1(X)$ and $\partial_X(\xi) = 0$.*

Proof. Decompose $\mathscr{M}_X$ as graded vector space by setting $F_0 = \Lambda(V_0)$ and, for each $k > 0$, $F_k = \bigoplus_{j=1}^{n} y_j F_{k-1}$. Note that $d_X(F_k) \subseteq F_{k-1}$ (where $F_{-1} = \{0\}$). For each $k \geq 0$, set

$$D_{0,k} = \mathrm{Span}\{P\partial x_i \mid P \in F_k, |P| < |x_i|,\ i \in \{1, \ldots, m\}\}$$

and

$$D_{1,k} = \mathrm{Span}\{P\partial y_j \mid P \in F_k, |P| < |y_j|,\ j \in \{1, \ldots, n\}\},$$

so that $D_{0,0} = D_0(X)$, $D_{1,0} = D_1(X)$ and $D_{1,1} = B(X)$. Using (2), we obtain

$$\partial_X(D_{0,k}) \subseteq D_{0,k-1} \oplus D_{1,k} \quad \text{and} \quad \partial_X(D_{1,k}) \subseteq D_{1,k-1}.$$

Write $\theta = \sum_{k \geq 0} \theta_{0,k} + \sum_{k \geq 0} \theta_{1,k}$ where $\theta_{i,k} \in D_{i,k}$. Let $\phi = \theta_{0,0} + \theta_{1,1}$ and $\xi = \theta - \phi$. Then $\partial_X(\theta) \in D_{1,0}$ implies $\partial_X(\xi) = 0$. $\square$

Finally, we show all boundaries in $B(X) \oplus D_0(X)$ vanish under p.

Lemma 4.6. *Let $\theta \in \mathrm{Der}_+(\mathscr{M}_X)$. If $\partial_X(\theta) \in B(X) \oplus D_0(X)$, then $p(\partial_X(\theta)) = 0$.*

Proof. Since $p(B(X)) = 0$, it suffices to assume $\partial_X(\theta) \in D_0(X)$. But in this case, $\partial_X(\theta) = \sum_{i=1}^{m} d_X(P_i)\partial x_i$ for some $P_i \in F_0 \oplus F_1$. Clearly, $p(\partial_X(\theta)) = 0$. $\square$

Lemmas 4.3–4.6 together imply Theorem 4.2. $\square$

5. Applications

Our most general result here is

Theorem 5.1. *Let X be any formal two-stage space. Then $\mathscr{L}_1(X)$ is an abelian ideal of $\pi_*(\Omega\mathrm{Baut}_1(X))$.*

Proof. We need only check that $D_1(X)$ is an abelian ideal of $B(X)\oplus D_0(X)\oplus D_1(X)$. Note that, if $P,Q\in\Lambda(V_0)$, then $[P\partial y_j, Q\partial y_k]=0$. Also,

$$[P\partial y_j, Q\partial x_i]=\pm Q\frac{\partial P}{\partial x_i}\partial y_j \quad\text{and}\quad [P\partial y_j, Qy_k\partial y_l]=\delta_{jk}PQ\partial y_l,$$

where δ_{jk} is the Kronecker delta. □

We now focus on the case of a product $X\times Y$ of two-stage spaces. Note that $\pi_*(\Omega\mathrm{Baut}_1(X))$ is naturally a subspace of $\pi_*(\Omega\mathrm{Baut}_1(X\times Y))$. We prove

Theorem 5.2. *Let X and Y be two-stage formal spaces with X an F_0-space satisfying Halperin's conjecture. Then*

$$\pi_*(\Omega\mathrm{Baut}_1(X))\subseteq\mathrm{center}(\pi_*(\Omega(\mathrm{Baut}_1(X\times Y))).$$

Proof. By Theorem 3.2, $\pi_*(\Omega\mathrm{Baut}_1(X))=\mathscr{L}_1(X)$. Since $\mathscr{L}_1(X\times Y)$ is abelian, it suffices to show $[\mathscr{L}_1(X),\mathscr{L}_0(X\times Y)]=0$. Write $(\mathscr{M}_Y,d_Y)=(\Lambda(W_0)\otimes\Lambda(W_1),d_Y)$, again for the Sullivan minimal model of Y. Choose bases $\{w_1,\ldots,w_s\}$ and $\{v_1,\ldots,v_t\}$ for W_0 and W_1, respectively. Let

$$D_0(X\times Y,Y)=\mathrm{Span}\{P\partial w_i\mid P\in\Lambda(V_0\oplus W_0)|P|<|w_i|\ i\in\{1,\ldots,s\}\}$$

and

$$B(X\times Y,Y)=\mathrm{Span}\{Pv_k\partial v_j\mid P\in\Lambda(V_0\oplus W_0),|P|+|v_k|<|v_j|,\ j,k\in\{1,\ldots,t\}\}.$$

By Theorems 3.3 and 4.2, the elements of $\mathscr{L}_0(X\times Y)$ are represented by derivations in $D_0(X\times Y,Y)\oplus B(X\times Y,Y)$. Such derivations clearly commute with those in $D_1(X)$. □

Theorem 5.3. *Let X and Y be as in Theorem 5.2. Then*

$$\mathrm{nil}(\pi_*(\Omega\mathrm{Baut}_1(X\times Y)))=\mathrm{nil}(\pi_*(\Omega\mathrm{Baut}(Y)))+\varepsilon,$$

where $\varepsilon=0$ or 1. If $\max(\pi_*(X))\le\min(\pi_*(Y))$ then $\varepsilon=0$.

Proof. We prove

$$\mathrm{nil}(\pi_*(\Omega\mathrm{Baut}_1(X\times Y))/\mathscr{L}_1(X\times Y))\le\mathrm{nil}(\pi_*(\Omega\mathrm{Baut}_1(Y))).$$

The first statement then follows from the fact that $\mathscr{L}_1(X\times Y)$ is an abelian ideal.

Consider the left $\Lambda(V_0)$-action on $\mathcal{M}_{X\times Y}$ induced by the isomorphism $\mathcal{M}_{X\times Y} \cong \mathcal{M}_X \otimes \mathcal{M}_Y$. It induces a left $\Lambda(V_0)$-action on $\mathrm{Der}_+(M_{X\times Y})$ after truncation in positive degrees. Similarly, the left $H^*(X)$-action on $H^*(X\times Y)$ induces one on $L(X\times Y)$. Observe that $p : \mathrm{Der}(\mathcal{M}_{X\times Y}) \to L(X\times Y)$ satisfies $p(P\theta) = \rho(P)p(\theta)$ where $\rho : \mathcal{M}_X \to H^*(X)$ is the formalization map.

The restriction of the $\Lambda(V_0)$-action to $D_0(X\times Y, Y) \oplus B(X\times Y, Y)$ satisfies

$$\text{(i) } \partial_{X\times Y}(P\theta) = P\partial_{X\times Y}(\theta) \quad \text{and} \quad \text{(ii) } [P\theta_1, Q\theta_2] = PQ[\theta_1, \theta_2]$$

for $P, Q \in \Lambda(V_0), \theta_1, \theta_2 \in D_0(X\times Y, Y) \oplus B(X\times Y, Y)$. The restriction of the $H^*(X)$-action to $L_0(X\times Y, Y)$ satisfies

$$\text{(iii) } D(\alpha \cdot z) = \alpha \cdot D(z)$$

for $\alpha \in H^*(X)$ and $z \in L_0(X\times Y, Y)$.

Suppose $z_1, z_2 \in \mathcal{L}_0(X\times Y)$ represent homogeneous elements with nontrivial bracket. By Theorem 3.3, $z_1, z_2 \in L_0(X\times Y, Y)$. For $i = 1, 2$, write $z_i = \sum_{j=1}^{d} \alpha_j z'_{ij}$, where $\{\alpha_1, \ldots, \alpha_d\}$ is a fixed additive basis for $H^*(X)$ and $z_{ij} \in L_0(Y)$. By (iii), $z'_{ij} \in \mathcal{L}_0(Y)$. Use Theorem 4.2 to choose representatives $\theta'_{ij} \in \mathcal{D}_0(Y)$ for z'_{ij}. Then $\theta_i = \sum_{j=1}^{d} P_j \theta'_{ij} \in \mathcal{D}_0(X\times Y)$ represents z_i where the $P_{ij} \in \Lambda(V_0)$ are cocycle representatives for the α_{ij}. By (i), $\theta'_{ij} \in \mathcal{D}_0(Y)$. Moreover, by (ii),

$$0 \neq p([\theta_1, \theta_2]) = \sum_{j=1}^{d} \sum_{k=1}^{d} \rho(P_j P_k) p([\theta'_{1j}, \theta'_{2k}]).$$

Thus there exist $\theta'_{1j}, \theta'_{2k} \in \mathcal{D}_0(Y)$ with $p([\theta'_{1j}, \theta'_{2k}]) \neq 0$ in $\mathcal{L}_0(Y)$.

For the second statement, note that the degree hypothesis implies

$$[D_1(X), D_0(X\times Y, Y) \oplus B(X\times Y, Y)] = 0. \qquad \square$$

When $H^*(Y)$ is free, we get the complete answer. Namely,

Theorem 5.4. *Let X be an F_0-space satisfying Halperin's conjecture and Y a nontrivial product of rational Eilenberg–Maclane spaces. Let $m = \max(\pi_*(Y))$ and $c = \mathrm{maxlen}(\pi_*(Y))$. Then*

$$\mathrm{center}(\pi_*(\Omega \mathrm{Baut}_1(X\times Y))) = \pi_*(\Omega \mathrm{Baut}_1(X)) \oplus \bigoplus_{k=1}^{m} H^{m-k}(X) \otimes \pi_m(Y)$$

and

$$\mathrm{nil}(\pi_*(\Omega \mathrm{Baut}_1(X\times Y))) = \begin{cases} c+1 & \text{if } \max(\pi_*(X)) > \max(\pi_*(Y)), \\ c & \text{if } \max(\pi_*(X)) \leq \max(\pi_*(Y)). \end{cases}$$

Proof. Here $W_1 = 0$ and $\Lambda(W_0) \cong H^*(Y)$. By Theorem 3.2,

$$\pi_*(\Omega \mathrm{Baut}_1(X\times Y)) = \mathcal{L}_1(X\times Y) \oplus \bigoplus_{n\geq 1} \bigoplus_{k\geq 0} H^k(X\times Y) \otimes W_0^{n+k}.$$

By Theorem 5.2, $\mathscr{L}_1(X) \subseteq \mathscr{L}_1(X \times Y)$ is central. For degree reasons, the subspace $\bigoplus_{k=1}^{m} H^{m-k}(X) \otimes W_0^m$ is central. Using cycle representatives, it is straightforward to check that there are no other central elements.

For the nilpotence result, let $S_1, \ldots, S_c$ denote a maximal elementary sequence for $\pi_*(Y)$, as defined in Section 1. Thus $S_1 = (|w_c|)$ for $w_c \in W_0$ of maximal degree. Each admissable sequence T gives rise to an element $P \in \Lambda(W_0)$ unique up to sign. Namely, P is the product of all w_i with $|w_i| \in T$ and 1 if T is empty. Let P_{c-j} be the polynomial and w_{c-j} be the basis element corresponding to the passage from S_j to S_{j+1}. Then

$$1\partial w_c = q \cdot [1\partial w_1[P_2\partial w_2[\cdots[P_{c-1}\partial w_{c-1}, P_c\partial w_c]\cdots]]]$$

represents a maximal nontrivial iterated bracket in $\pi_*(\Omega \mathrm{Baut}_1(Y))$ of length c.

If $\max(\pi_*(X)) > \max(\pi_*(Y))$ choose $y_j \in V_1$ with $|y_j| > |w_c|$. The elements $1\partial y_j$ and $w_c\partial y_j$ in $D_1(X \times Y)$ clearly cannot bound. Thus,

$$1\partial y_j = q \cdot [1\partial w_1[P_2\partial w_2[\cdots[P_{c-1}\partial w_{c-1}[P_c\partial w_c, w_c\partial y_j]]\cdots]]]$$

represents an nontrivial iterated bracket of length $c+1$. The first case follows from Theorem 5.3.

Finally, since $\mathscr{L}_1(X \times Y)$ is an abelian ideal, any nontrivial iterated bracket in $\pi_*(\Omega \mathrm{Baut}_1(X \times Y))$ can involve at most one element from this subspace and this element must occur at the innermost bracket. The nilpotence when $\max(\pi_*(X)) \leq \max(\pi_*(Y))$ follows easily. $\square$

5.5. *Remarks*

If F is pure, elliptic and formal, then [4] implies $F \simeq_{\mathbb{Q}} X \times Y$, where X is an F_0-space and Y is a product of odd spheres. Theorem 5.4 thus computes the center and nilpotence of the rational homotopy Lie algebra of the classifying space for all pure, elliptic, formal spaces, modulo the Halperin conjecture.

By Salvatore [10, Theorem 3], $\mathrm{Hnil}_0(\mathrm{aut}_1(X)) = \mathrm{nil}(\pi_*(\Omega \mathrm{Baut}_1(X)))$. Examples 1.1(1) and (2) thus follow from Theorem 5.4 and the proof of Halperin's conjecture for truncated polynomial algebras [8]. Example 1.1(3) follows from the main result of Shiga and Tezuka [12] together with the work of Borel in [2], which implies rational factorizations

$$U(n)/U(n_1) \times \cdots \times U(n_k) \simeq_{\mathbb{Q}} U(m)/U(n_1) \times \cdots \times U(n_k) \times S^{2(m+1)-1} \times \cdots \times S^{2n-1},$$

$$Sp(n)/Sp(n_1) \times \cdots \times Sp(n_k) \simeq_{\mathbb{Q}} Sp(m)/Sp(n_1) \times \cdots \times Sp(n_k) \times S^{4(m+1)-1} \times \cdots \times S^{4n-1}.$$

Example 1.1(4) follows from the main result of Lupton [7].

In [15], we use the techniques of this paper to determine the full rational homotopy type of $\mathrm{Baut}_1(X \times Y)$ under further restrictions on the spaces X and Y.

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